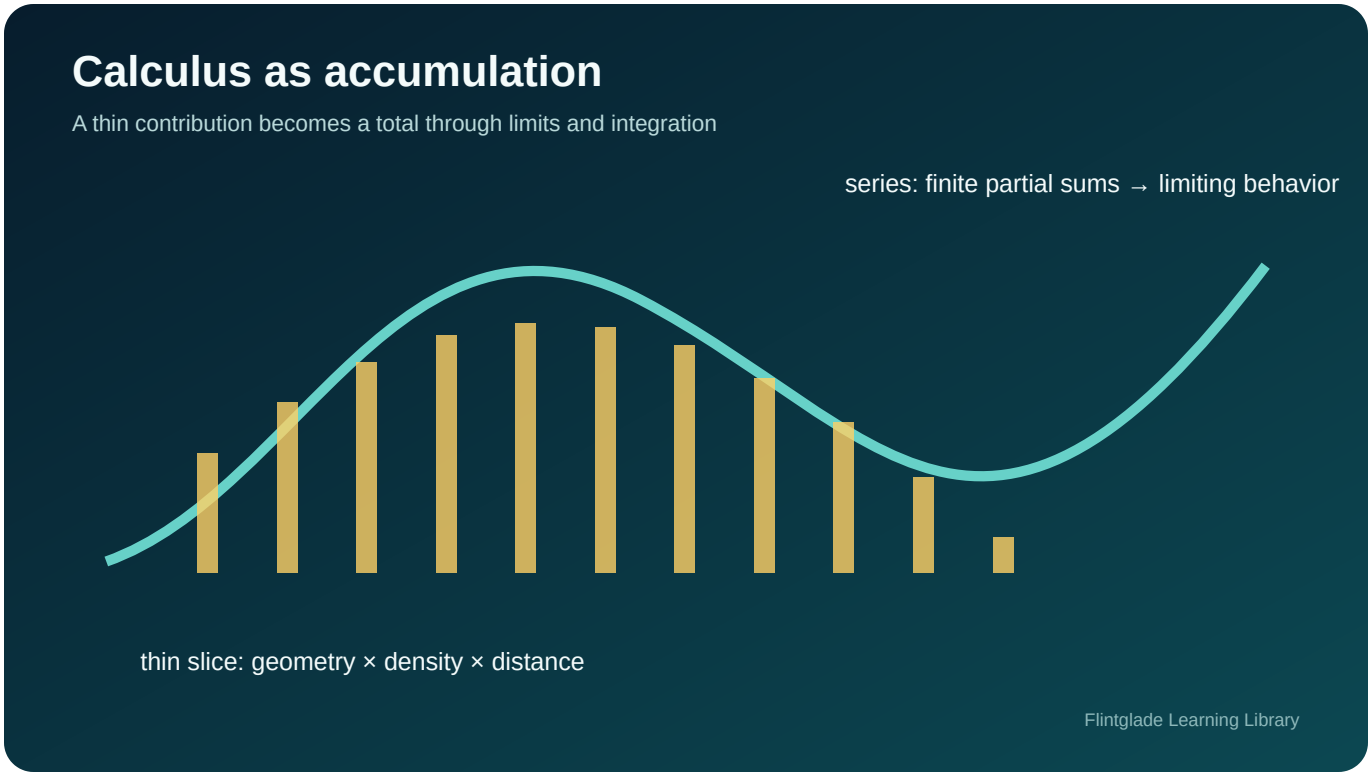
Calculus II: Integration Applications and Series

Flintglade Learning Library · Calculus

Audience: Calculus learners studying work, volumes, arc length, sequences, and convergence methods.



About this edition: This standalone edition organizes the material for independent reading with clear navigation, reusable reasoning frameworks, and print-ready presentation.

Learning goals

- Translate physical and geometric situations into integrals
- Analyze sequences and recognize common series structures
- Select and justify an appropriate convergence method

Navigate this guide

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Mathematical Setup Before Calculation

Many Calculus II problems are difficult because the quantity to integrate is not given directly. The central skill is modeling: choose a thin piece, describe its contribution, and add those contributions with an integral.

Use this pattern:

1. Draw or describe the geometry.
2. Choose a variable and an interval.
3. Express every changing quantity using that variable.
4. Write the contribution of one thin slice.
5. Integrate over the full interval.
6. Check units, sign, and scale.

The same habit carries from work and volume into arc length and later into series estimates: define what accumulates before applying a formula.

Work from a Variable Force

For a constant force parallel to motion, work is force times distance. When force changes with position,

$$W = \int_a^b F(x)\,dx.$$

The units should be force multiplied by distance, such as newton-meters or foot-pounds.

Lifting a hanging cable

Suppose a 12-meter cable has linear mass density 3 kg/m and hangs vertically from a platform. Ignore stretching. A small piece at depth x has mass $3\,dx$ and weight $3g\,dx$. It must be lifted distance x, so

$$dW = (3g\,dx)x.$$

Therefore,

$$W = \int_0^{12} 3gx\,dx = \frac{3g}{2}(12)^2.$$

With $g = 9.8\ \text{m/s}^2$, this is $2116.8\ \text{J}$.

This model can also be derived by observing that the amount of cable still hanging changes as the cable rises. Both views agree when the geometry and limits are consistent.

Pumping a Fluid

A horizontal fluid slice contributes

$$dW = (\text{weight density})(\text{slice volume})(\text{lifting distance}).$$

If $A(y)$ is the cross-sectional area at height y, then slice volume is $A(y)\,dy$. For fluid weight density ρ_w and outlet height H,

$$W = \int_a^b \rho_w A(y)(H - y)\,dy.$$

Worked example: an inverted conical tank

An inverted cone is 6 m tall with top radius 3 m. Water fills the cone to depth 4 m and is pumped to an outlet 1 m above the top. Let $y = 0$ at the tip and measure upward.

Similar triangles give

$$\frac{r(y)}{y} = \frac{3}{6} = \frac{1}{2}, \qquad r(y) = \frac{y}{2}.$$

Thus,

$$A(y) = \pi r(y)^2 = \frac{\pi y^2}{4}.$$

The outlet is at height 7, so a slice travels $7 - y$. Using water's weight density $9800\ \text{N/m}^3$,

$$W = 9800 \int_0^4 \frac{\pi y^2}{4}(7 - y)\,dy.$$

The setup is the main modeling result. Expanding gives

$$W = 2450\pi \int_0^4 (7y^2 - y^3)\,dy = 2450\pi \left[\frac{7y^3}{3} - \frac{y^4}{4}\right]_0^4.$$

Partial filling and partial pumping

The integration interval describes the fluid that actually moves. If water occupies only part of a tank, change the bounds. If pumping stops at a lower outlet or after only the upper layer is removed, adjust both the interval and lifting distance. Re-deriving these two features is safer than adapting a remembered setup.

Volumes of Revolution

The disk/washer method slices perpendicular to the axis of rotation:

$$V = \pi \int_a^b (R(x)^2 - r(x)^2)\,dx.$$

The shell method slices parallel to the axis:

$$V = 2\pi \int_a^b (\text{radius})(\text{height})\,dx.$$

Choose the method that describes the region with fewer pieces or fewer inverse functions.

Worked example: shells around a vertical line

Rotate the region beneath $y = 4 - x^2$ and above $y = 0$ for $0 \le x \le 2$ about the line $x = -1$. A vertical strip forms a shell with

$$\text{radius} = x + 1, \qquad \text{height} = 4 - x^2.$$

Hence

$$V = 2\pi \int_0^2 (x + 1)(4 - x^2)\,dx.$$

The radius is a distance, so it must be nonnegative. Writing it explicitly as the distance from x to the axis prevents a common sign error.

Arc Length

For a differentiable graph $y = f(x)$ on $[a, b]$, approximate the curve by short line segments. A small change has horizontal component dx and vertical component $dy = f'(x)dx$. The Pythagorean theorem gives

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + [f'(x)]^2}\, dx.$$

Therefore,

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

If $x = g(y)$ is simpler,

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

Arc-length function

Fix a starting coordinate a. The accumulated length from a to x is

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt.$$

The dummy variable t distinguishes the integration variable from the upper limit. By the Fundamental Theorem of Calculus,

$$s'(x) = \sqrt{1 + [f'(x)]^2}.$$

Worked example with a designed simplification

Let

$$f(x) = \frac{x^2}{4} - \frac{1}{2}\ln x, \qquad 1 \le x \le 2.$$

Then

$$f'(x) = \frac{x}{2} - \frac{1}{2x}.$$

Now simplify the radicand before choosing a substitution:

$$1 + [f'(x)]^2 = 1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2 = \left(\frac{x}{2} + \frac{1}{2x}\right)^2.$$

Because $x > 0$ on the interval,

$$\sqrt{1 + [f'(x)]^2} = \frac{x}{2} + \frac{1}{2x}.$$

Thus

$$L = \int_1^2 \left(\frac{x}{2} + \frac{1}{2x}\right)dx = \left[\frac{x^2}{4} + \frac{1}{2}\ln x\right]_1^2 = \frac{3}{4} + \frac{1}{2}\ln 2.$$

Always inspect $1 + (f')^2$ for a perfect square. Arc-length functions are often chosen so that this expression simplifies.

Sequences

A sequence is a function whose inputs are positive integers. We write its terms as $a_1, a_2, a_3, \ldots$ and study what happens as n grows.

The sequence converges to L if

$$\lim_{n \to \infty} a_n = L.$$

Otherwise it diverges. Divergence can arise from unbounded growth, persistent oscillation, or any failure to approach one finite value.

Dominant growth

For rational expressions in n, compare highest-order terms. For example,

$$a_n = \frac{5n^3 - 2n}{2n^3 + 7}.$$

Divide numerator and denominator by n^3:

$$a_n = \frac{5 - 2/n^2}{2 + 7/n^3} \to \frac{5}{2}.$$

Logarithms grow more slowly than positive powers, powers grow more slowly than exponentials with base greater than one, and exponentials grow more slowly than factorials. This hierarchy is a guide, not a substitute for checking signs and domains.

Oscillation with shrinking magnitude

The factor $(-1)^n$ does not force divergence. Magnitude matters:

$$a_n = \frac{(-1)^n}{n} \to 0,$$

whereas $(-1)^n$ alone has no limit.

Infinite Series and Partial Sums

An infinite series

$$\sum_{n=1}^{\infty} a_n$$

is defined through its partial sums

$$S_N = \sum_{n=1}^{N} a_n .$$

The series converges if the sequence S_N approaches a finite limit. This distinction matters: $a_n \to 0$ is necessary for convergence of the series, but it is not sufficient.

Geometric Series

A geometric series has a constant ratio:

$$\sum_{n=0}^{\infty} ar^n .$$

It converges when $|r| < 1$ and then

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

If the index begins elsewhere, identify the actual first term rather than applying the formula to the coefficient mechanically.

Worked example

Consider

$$\sum_{n=2}^{\infty} 6\left(-\frac{1}{3}\right)^n .$$

The first included term is $6(1/9) = 2/3$, and the ratio is $-1/3$. Therefore,

$$\sum_{n=2}^{\infty} 6\left(-\frac{1}{3}\right)^n = \frac{2/3}{1-(-1/3)} = \frac{1}{2}.$$

Telescoping Series

A telescoping series exposes cancellation after partial fractions or another rewrite. For example,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}.$$

Since

$$\frac{1}{n(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right),$$

the partial sum is

$$S_N = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right).$$

Taking the limit gives

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{3}{4}.$$

Write several terms or the finite partial sum. Saying “everything cancels” can hide the boundary terms that determine the value.

The Divergence Test

If

$$\lim_{n \to \infty} a_n \neq 0$$

or the limit does not exist, then $\sum a_n$ diverges. If the limit is zero, this method is inconclusive. The harmonic series $\sum 1/n$ is the standard reminder: its terms approach zero, but its partial sums grow without bound.

Use this check early because it can settle divergence quickly, but do not use it to claim convergence.

Integral Test and Remainder Bounds

Suppose $a_n = f(n)$ where f is positive, continuous, and decreasing for sufficiently large x. Then

$$\sum_{n=N}^{\infty} a_n$$

and

$$\int_N^{\infty} f(x)\,dx$$

either both converge or both diverge.

For a convergent series, if $R_N = S - S_N$, then

$$\int_{N+1}^{\infty} f(x)\,dx \leq R_N \leq \int_N^{\infty} f(x)\,dx.$$

Worked example: a p-series tail

For $\sum_{n=1}^{\infty} 1/n^3$,

$$R_N \le \int_N^\infty x^{-3}\,dx = \frac{1}{2N^2}.$$

To ensure the remainder is at most 0.0005, require

$$\frac{1}{2N^2} \le 0.0005,$$

which gives $N^2 \ge 1000$. Thus $N = 32$ is sufficient.

Direct Comparison

For nonnegative terms:

- If $0 \le a_n \le b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $0 \le b_n \le a_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

The inequality must point in a useful direction. A smaller series than a divergent benchmark tells us nothing, and a larger series than a convergent benchmark tells us nothing.

Worked example

For $n \ge 1$,

$$0 \le \frac{1}{n^2 + 4n + 1} \le \frac{1}{n^2}.$$

Because $\sum 1/n^2$ converges, the given series converges by direct comparison.

Limit Comparison

For positive terms a_n and b_n, if

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c$$

with $0 < c < \infty$, the two series have the same convergence behavior.

Choose b_n from dominant terms. For

$$a_n = \frac{3n + 5}{n^3 - 2},$$

the dominant behavior is $3n/n^3 = 3/n^2$, so $b_n = 1/n^2$ is natural:

$$\lim_{n \to \infty} \frac{a_n}{1/n^2} = \lim_{n \to \infty} \frac{3n^3 + 5n^2}{n^3 - 2} = 3.$$

The original series therefore converges.

Alternating Series

An alternating series can often be written

$$\sum_{n=N}^{\infty} (-1)^n b_n$$

or with $(-1)^{n+1}$, where $b_n \geq 0$. It converges if b_n eventually decreases and $b_n \to 0$.

If those conditions hold, the truncation error satisfies

$$|R_N| \leq b_{N+1}.$$

This gives a practical stopping rule. It does not by itself establish absolute convergence; analyze $\sum b_n$ separately when that distinction matters.

Selecting a Convergence Method

Use structural clues:

1. **Term limit:** If a_n does not approach zero, divergence is settled.
2. **Geometric form:** Look for a constant ratio and use $|r| < 1$.
3. **Telescoping form:** Try partial fractions when shifted factors suggest cancellation.
4. **Positive rational or radical terms:** Compare dominant powers with a p-series.
5. **A convenient positive decreasing function:** Consider the integral method, especially when a remainder estimate is needed.
6. **Alternating signs:** Check decreasing magnitude and a zero limit; then consider absolute convergence separately.

No decision tree replaces algebraic inspection. Simplifying the term can reveal a structure that was initially hidden.

Common Modeling and Reasoning Errors

- Mixing coordinate systems midway through a work integral.
- Using total fluid volume instead of a thin slice volume.
- Measuring shell radius from the vertical axis when rotation is about a shifted line.
- Forgetting the square root in the arc-length formula.
- Treating $a_n \to 0$ as sufficient for convergence of $\sum a_n$.
- Comparing in an unhelpful inequality direction.
- Omitting positivity or eventual decrease when using methods that require them.
- Applying a geometric sum formula without identifying the first included term.
- Reporting a decimal without checking whether the scale and units are plausible.

Compact Reference

Topic	Core expression	Essential check
Variable work	$W = \int F(x)\,dx$	Force and displacement use one variable
Fluid pumping	$W = \int \rho_w A(y)(H-y)\,dy$	Bounds match the fluid that moves
Shell volume	$V = 2\pi \int (\text{radius})(\text{height})$	Radius is distance to the rotation line
Arc length	$L = \int \sqrt{1+(f')^2}$	Simplify the radicand first
Geometric series	$a/(1-r)$	$
Divergence check	$a_n \not\to 0 \Rightarrow \sum a_n$ diverges	Zero limit is inconclusive
Integral remainder	$\int_{N+1}^{\infty} f \le R_N \le \int_N^{\infty} f$	Positive, continuous, decreasing
Limit comparison	$a_n/b_n \to c$	$0 < c < \infty$
Alternating remainder	$	R_N

The unifying idea is accumulation under explicit assumptions. A correct integral or series argument names what is being accumulated, why the formula applies, and how the result can be checked.